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Transient response with arbitrary initial conditions using the DFT

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Abstract

It is more economic to compute the response of linear systems with Fourier methods using fast Fourier transform algorithms than with step-by-step numerical integration methods. However, one drawback of Fourier methods is the difficulty in computing transient responses with arbitrary initial conditions (ICs). When the system is modeled with constant-parameter ordinary differential equations, the response can be obtained in closed form but, when using spectral and boundary element methods, this is no longer possible. In this paper, a technique consisting of taking advantage of the periodic character of the discrete Fourier transform to include an ad hoc force pulse to impose the ICs is proposed. The technique is presented in detail and used to compute the responses of single and multiple degree-of-freedom lumped parameter systems. The responses are compared with step-by-step integration solutions.

1. Introduction

The discrete Fourier transform (DFT), efficiently computed by fast algorithms, e.g. the fast Fourier transform (FFT) [1], is largely used for predicting the steady-state periodic response of linear dynamical systems. It is also possible, although not so usual, to compute the transient response of damped linear systems with the DFT for null initial conditions (ICs).

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For linear systems that can be described by time-domain, constant-parameter ordinary differential equations, it is possible to combine the forced response for null ICs with the analytical solution for arbitrary ICs after decoupling the system of equations by modal analysis [2]. However, when the dynamic equations are only known in the frequency domain, as it is the case when using spectral methods [3] and boundary element methods [4], this is no longer possible.

In this paper, the possibility of using the DFT to predict the transient response of damped linear time-invariant systems with arbitrary ICs is investigated. The proposed method consists of introducing a previous impulsive force which drives the system to the initial conditions at the initial time, taking advantage of the periodic character of the DFT, which is characterized by Poisson's formula [5]. The methodology is applied to a lumped parameter, multi-degree-of-freedom (mdof) mechanical system for which the response can be compared with a step-by-step integration solution.

2. Transient response via the DFT

The DFT of a sampled signal $x_n = x(t = n\,\Delta t)$ may be defined as [1]

$$X_k = \frac{1}{N}\sum_{n=0}^{N-1} x_n \mathrm{e}^{-\mathrm{i}2\pi kn/N}, \quad k = 0, \ldots, N-1 \tag{1}$$

and, correspondingly,

$$x_n = \sum_{k=0}^{N-1} X_k \mathrm{e}^{\mathrm{i}2\pi kn/N}, \quad n = 0, \ldots, N-1. \tag{2}$$

Using the sampling theorem and taking an integer number of periods of a periodic signal in the observation window, it can be shown [5] that the DFT produces exactly the Fourier series coefficients of the signal. The DFT can also be used to calculate the Fourier integral for transient signals. Poisson's formula relates the Fourier series and Fourier integral [5]:

$$\bar{x}(t) = \sum_{n=-\infty}^{+\infty} x(t + nT) = \sum_{k=-\infty}^{+\infty} \frac{X(f = k/T)}{T}\, \mathrm{e}^{\mathrm{i}2\pi kt/T}, \tag{3}$$

where

$$X(f) = \int_{-\infty}^{+\infty} x(t)\mathrm{e}^{-\mathrm{i}2\pi ft}\,\mathrm{d}t. \tag{4}$$

Poisson's formula shows that, when the DFT is used, the transient signal is made periodic with a period equal to the observation time $T = N\,\Delta t$ and the Fourier series coefficients of the *periodized* signal $\bar{x}(t)$, X_k, are related to the Fourier integral of the original signal $x(t)$, $X(f)$, by

$$X_k = \frac{X(f = k/T)}{T}. \tag{5}$$

The DFT of a transient signal will be related to its Fourier integral by Eq. (5), provided that the transient vanishes within the observation time T. The physical interpretation of that mathematical property in the case of dynamic responses is that the response of a damped linear system to a transient is equal to the steady-state response of the system to the periodic transient, provided that the response practically vanishes within the period T. Mathematically, the damped response only vanishes for $t \to \infty$ but, for practical proposes, it is sufficient that the response $|u(t)| < \varepsilon; t > T$, with ε being an error tolerance.

The input/output relation for linear time-invariant systems may be expressed in terms of the frequency response function $H(f)$:

$$U(f) = H(f)F(f), \tag{6}$$

where

$$u(t) \overset{\mathscr{F}}{\longleftrightarrow} U(f), \quad f(t) \overset{\mathscr{F}}{\longleftrightarrow} F(f), \tag{7,8}$$

with $\overset{\mathscr{F}}{\longleftrightarrow}$ denoting the Fourier Transform or Fourier series, whichever is applicable, $f(t)$ being the input signal and $u(t)$ the output signal.

For discrete frequencies $f = k/T$:

$$U_k = H_k F_k, \quad k = 0, \ldots, \frac{N}{2} - 1. \tag{9}$$

The periodic nature of the DFT and the fact that the input and output signals are real in the time domain lead to the relations

$$U_k = \mathrm{Real}(U_k), \quad k = 0 \quad \text{and} \quad k = \frac{N}{2}, \tag{10}$$

$$U_k = U^*_{N-k}, \quad k = 1, \ldots, \frac{N}{2} - 1, \tag{11}$$

where U^* denotes the complex conjugate of U. Similar relations can be written for F_k. These symmetry properties imply that the DC and Nyquist terms, U_0 and $U_N/2$, must be real.

The periodicity of the DFT is purely mathematical and does not have any physical meaning.

H_k is not defined for $k > N/2$ and U_k should be calculated only for $k = 0, N/2$. But, in order to have a real response $u(t)$, it is necessary to "build" the symmetric part using Eqs. (10) and (11) before computing the inverse DFT.

Summarizing, in order to calculate the transient response via the DFT, it is necessary to choose an observation time T within which the response practically vanishes, and to enforce the symmetric characteristic of the DFT of real periodic signals before performing the inverse DFT.

3. Transient response with nonzero ICs

The problem of transient response calculation with nonzero ICs reduces to the problem of calculating a previous transient input which will drive the system to the ICs after a certain time t_p. Different impulsive shapes can be chosen and, given a shape, the appropriate amplitude, duration and time delay to reach the desired IC must be determined.

The response to the previous transient may be obtained by analytical or numerical methods. Convolution with simple shapes, e.g., rectangular and triangular pulses, is easy to calculate analytically, but this kind of shape is not suitable for the subsequent DFT application because of the unavoidable aliasing due to the poor sampling of the edges. For this reason, a smooth pulse, such as $A - A\cos(\omega t), t = [0, \pi/\omega)$ (shown in Fig. 1), must be used as previous transient input.

Among the numerical methods that can be used to compute the response of linear systems to the previous pulse—state-space methods, discrete convolution methods, step-by-step numerical integration methods and discrete Fourier Transform methods—the latter are the natural choice in this context.

The DFT is applied to the discrete transient input signal $p(t)$, which is a vector with N elements p_i, with $p_i = p(t = i\Delta t)$, where Δt is the sampling time interval. The previous transient that will lead the system to the desired IC will be placed at the tail of the discrete excitation vector, and, because of the periodic effect of the DFT, it will precede the initial time $t_0 = 0$, which corresponds to the ICs.

As the system is linear, the response to the previous transient may be tabulated so that it is not necessary to recalculate it for different ICs for the same system.

For single degree-of-freedom (dof) systems, the method is very simple. A time when the system has the same ratio (within a given accuracy) of displacement over velocity as that of the desired ICs may be found, and the previous transient signal must be shifted and scaled using this information.

For mdof systems the process is slightly more complicated, due to the influence of all input dofs over each output, but a solution exists provided that the number of transient inputs equals the number of outputs.

3.1. Formulation for single-dof systems

To apply this methodology to a single-dof system, it is necessary to perform the steps described below:

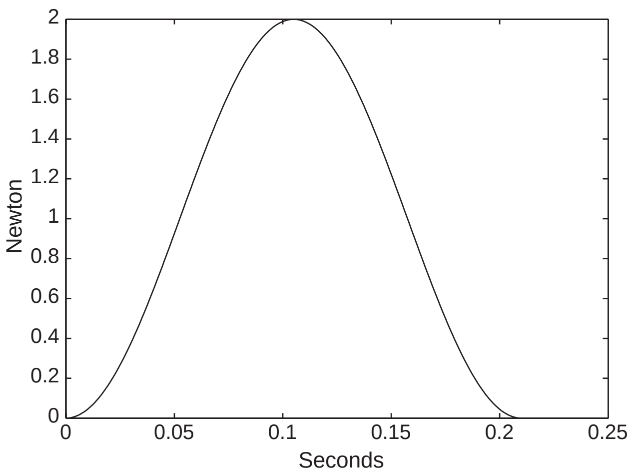

Fig. 1. Previous transient force shape.

(1) Choose a discrete transient signal shape p, where

$$p_i \begin{cases} \neq 0, & i = 0, \ldots, n_p - 1, \\ = 0, & i = n_p, \ldots, N - 1, \end{cases} \tag{12}$$

with $n_p \ll N$, typically $n_p/N < 0.1$, and $t_p = n_p \Delta t$.

(2) Calculate the response of the linear system under investigation to the transient p, x_i and $\dot{x}_i$, $i = 0, \ldots, N-1$, using the DFT (as explained in the previous section).

(3) Tabulate the results as the quotients between the response signal and its derivative (or vice-versa) versus time, i.e. $\dot{x}_i/x_i$ (or $x_i/\dot{x}_i$).

(4) Choose the instant $t_k = k\Delta t$ at which the response approximately reaches the desired IC quotient $\dot{x}_k/x_k \simeq \dot{u}_0/u_0$ (or $x_k/\dot{x}_k \simeq u_0/\dot{u}_0$). It is interesting to see that, theoretically, any relationship can be found in a half period of the response, but, in practice, it is not always possible to attain the desired relation within a given precision due to the discretization of the signals.

(5) Calculate the scaling factor C, given by $C = u_0/x_k$ or $C = \dot{u}_0/\dot{x}_k$. Both quotients must give the same result, since $u_0/\dot{u}_0 \simeq x_k/\dot{x}_k$.

(6) Build the previous excitation signal g which is given by

$$g_i = \begin{cases} 0, & i = 0, \ldots, N-k-1, \\ Cp_{i-(N-k)}, & i = N-k, \ldots, N-1. \end{cases} \tag{13}$$

(7) Superpose the previous excitation signal to the discretized excitation transient input $f(t)$ which must have been sampled with the same sampling interval Δt. If f is the sampled transient, the

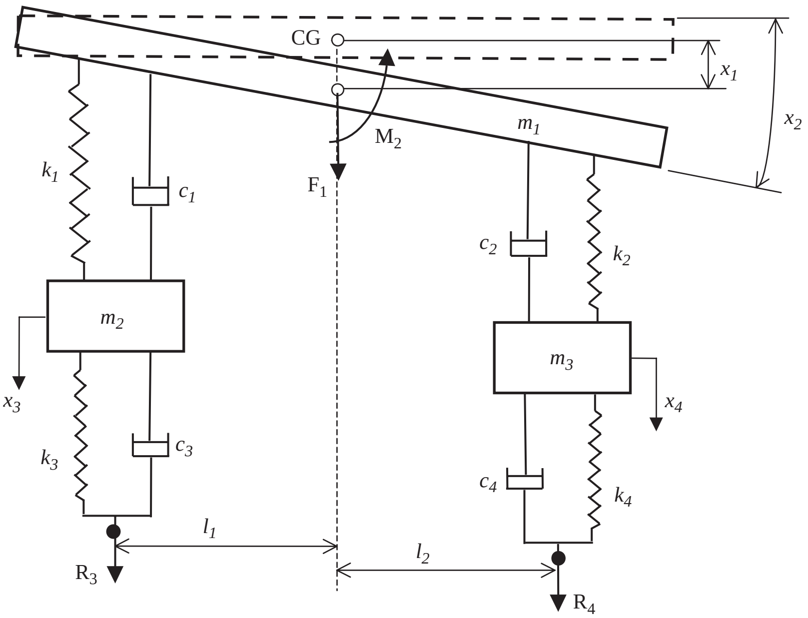

Fig. 2. Four dof model.

total resultant input vector fr will be

$$\mathrm{fr}_i = f_i + g_i, \quad i = 0, \ldots, N-1. \tag{14}$$

The frequency domain response to the input f and with the desired IC can be now be calculated using the FRF of the system:

$$U_k = H_k \mathrm{Fr}_k, \tag{15}$$

Table 1
Model parameters

$m_1 = 1461.8\,\mathrm{kg}$	$k_1 = 35016.4\,\mathrm{N/m}$	$c_1 = 1750.8\,\mathrm{N\,s/m}$
$m_2 = 10.0\,\mathrm{kg}$	$k_2 = 37934.4\,\mathrm{N/m}$	$c_2 = 1896.7\,\mathrm{N\,s/m}$
$m_3 = 10.0\,\mathrm{kg}$	$k_3 = 350164.0\,\mathrm{N/m}$	$c_3 = 17508\,\mathrm{N\,s/m}$
$I = 2176.2\,\mathrm{kg\,m^2}$	$k_4 = 379344.0\,\mathrm{N/m}$	$c_4 = 18967\,\mathrm{N\,s/m}$
$l_1 = 1.37\,\mathrm{m}$	$l_2 = 1.68\,\mathrm{m}$	

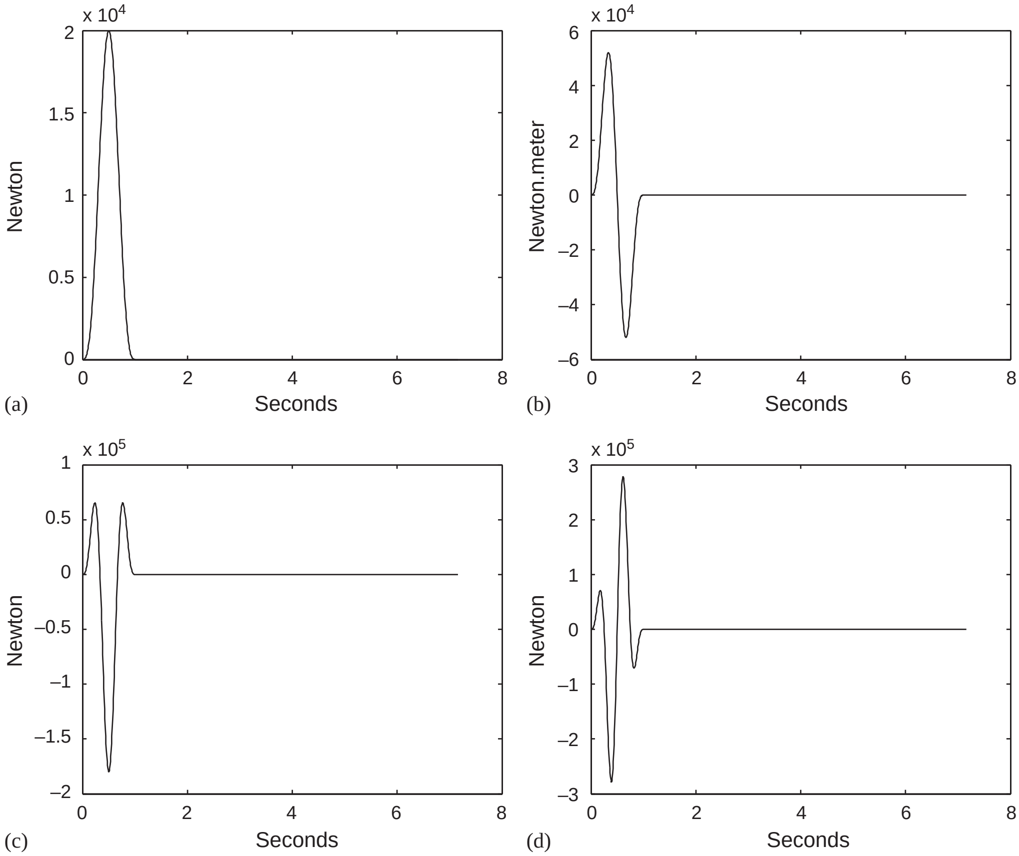

Fig. 3. Transient forces: (a) F_1, (b) M_2, (c) F_3, (d) F_4.

where Fr is the DFT of fr. Using the symmetry property in Eqs. (10) and (11) and the inverse DFT of U, it is finally possible to obtain the response in the time domain.

3.2. Formulation for mdof systems

For mdof systems, the desired transient response to given ICs must be constructed by shifting each transient response corresponding to a particular dof by a different amount of time and scaling it conveniently. Next, the methodology is described in more detail for a linear system with M inputs and M outputs (M dofs).

(1) Choose a previous discrete transient signal shape p, where

$$p_i \begin{cases} \neq 0, & i = 0, \ldots, n_p - 1, \\ = 0, & i = n_p, \ldots, N - 1, \end{cases} \tag{16}$$

with $n_p \ll N$, typically $n_p/N < 0.1$. At this step, the transient excitation p_i can be applied to any arbitrary dof m, provided that the response is not null for the DOFs where nonzero ICs are to be imposed.

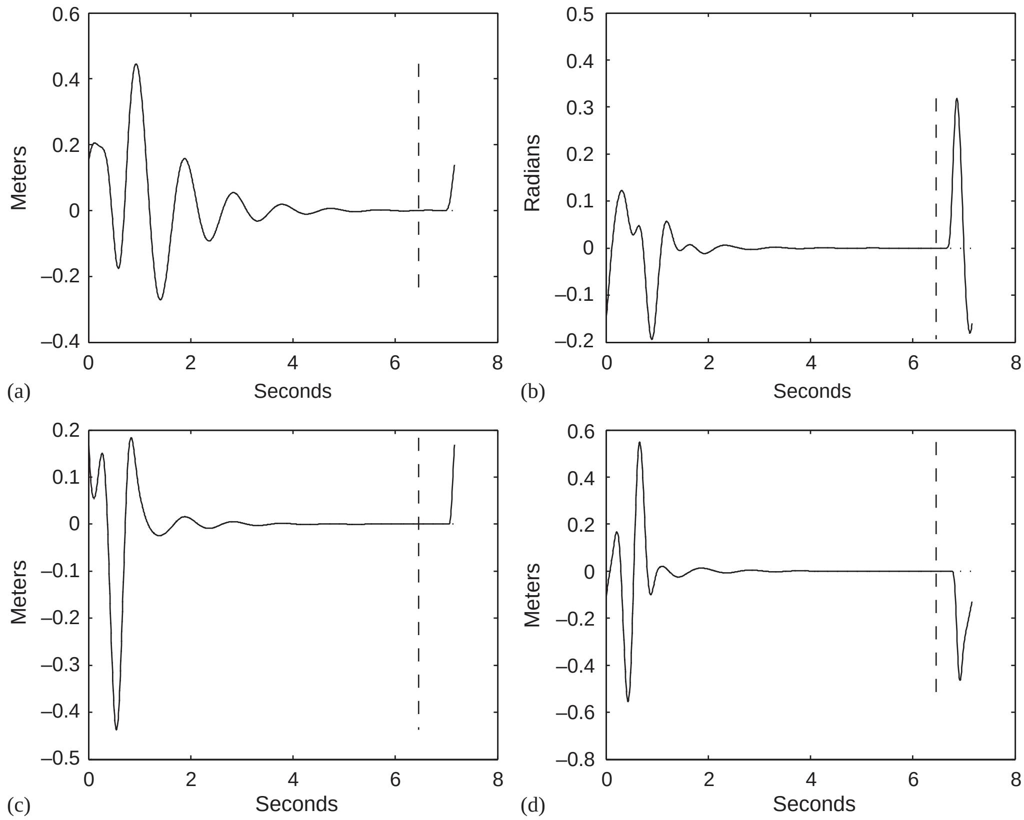

Fig. 4. Responses for example 1: (—) proposed method; (···) Newmark integration. (a) x_1, (b) x_2, (c) x_3, (d) x_4.

(2) Calculate the response of the linear system under investigation, x_{il} and $\dot{x}_{il}, i = 0, \ldots, N-1$; $l = 1, \ldots, M$, using the DFT and the FRF (as explained in the previous section), to the previous transient signal p, applied at dof m.
(3) Tabulate the results as the quotient between the response signal and its derivative (or vice-versa) versus time, i.e., $\dot{x}_{il}/x_{il}$ (or $x_{il}/\dot{x}_{il}$).
(4) Choose the instant $t_{kl} = k_l \Delta t$ at which the response approximately reaches the desired IC quotient $\dot{u}_{0l}/u_{0l}$ (or $u_{0l}/\dot{u}_{0l}$).
(5) Calculate the scaling factor C_l, given by $C_l = u_{0l}/x_{kl}$ or $C_l = \dot{u}_{0l}/\dot{x}_{kl}$. Both quotients must give approximately the same result, due to $u_{0l}/\dot{u}_{0l} \approx x_{kl}/\dot{x}_{kl}$.
(6) Shift each response by a time $t_l = k_l \Delta t$ to the left. Now the instant $t = 0$ is the instant when the relationship $\dot{x}_l/x_l$ (or $x_l/\dot{x}_l$) has the same value of the relation of the desired IC.
(7) Multiply each response by the corresponding scaling factor C_l.
(8) Transform each response to the frequency domain, using the DFT.
(9) Multiply by the inverse frequency response matrix of the system. The signals obtained are the desired previous transient excitation signals, in the frequency domain.
(10) Apply the inverse DFT to each of the previous signals, obtaining the desired previous transient excitation signals in the time domain, g_{il}.

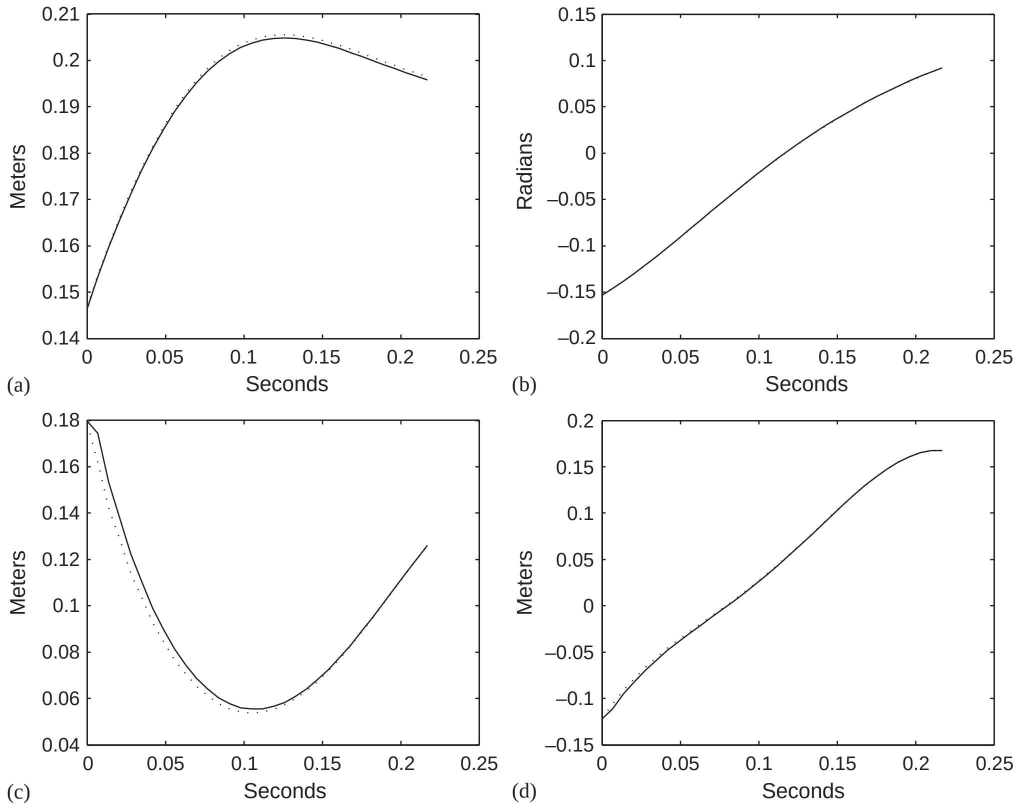

Fig. 5. Detail of responses for example 1: (—) proposed method; (···) Newmark integration. (a) x_1, (b) x_2, (c) x_3, (d) x_4.

(11) Erase the computed input signals leaving only the tail at the end of the block, because no perturbation must be done by the previous transient after the desired IC are attained. Note that, due to causality, the previous transient force before the shifted part at the end of the signals is null. A detailed analysis of the term g_{0l} will be made later in this section.

(12) Superpose the previous excitation signals to the discretized excitation transient input $f_l(t)$, which must have been sampled with the same sampling interval Δt. If f is the sampled transient, the total resultant input vector fr will be

$$\mathrm{fr}_{il} = f_{il} + g_{il}, \quad i = 0, \ldots, N-1; \quad l = 1, \ldots, M. \tag{17}$$

The response of the system in the time domain, with the desired ICs, can now be calculated as explained in the previous section.

Note that the proposed technique assumes that the original transient excitation and the response after $t = 0$ are not affected by the introduction of the previous transient force. Then, the introduced previous transient force must begin after the end of the original transient and only when responses have virtually vanished, and finish by the end of the time observation window. On

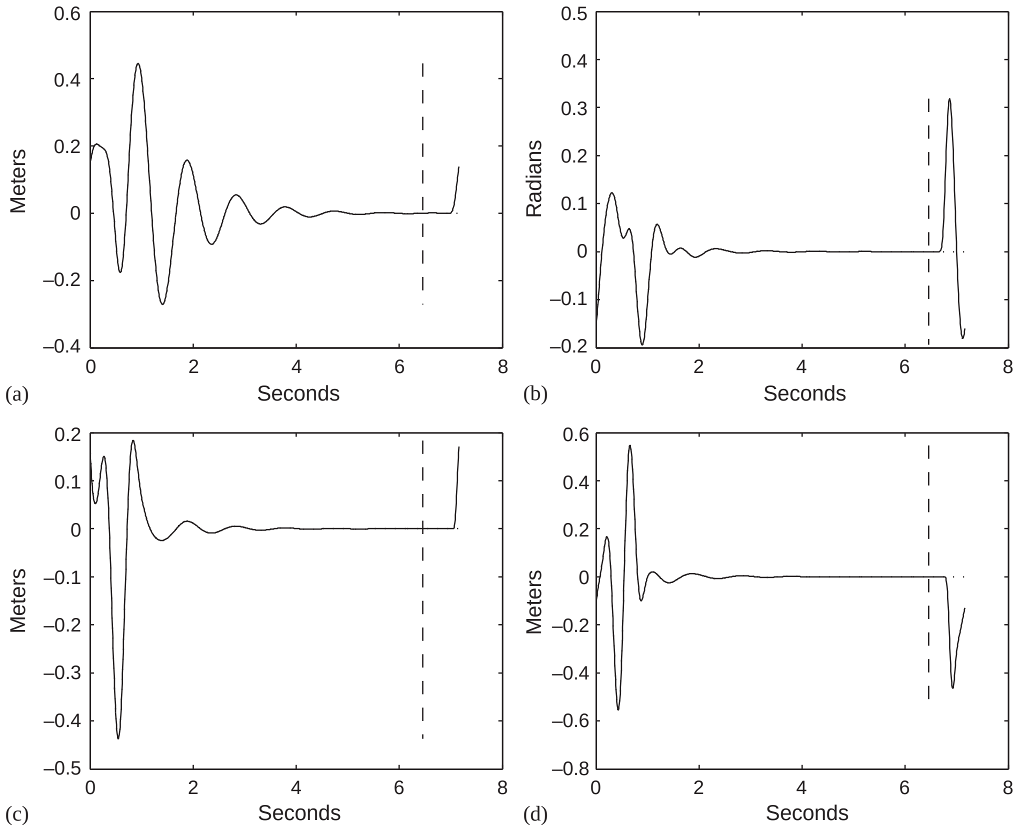

Fig. 6. Responses for example 2: (—) proposed method; (···) Newmark integration. (a) x_1, (b) x_2, (c) x_3, (d) x_4.

the other hand, the desired ICs are only attained if the previous transient force acts exactly until $t = 0$. However, in discrete systems, any force f_k has the effect of a constant force acting in a time interval Δt, centered at $t = k\,\Delta t$. Then, if g_{0l} is not zero, as needed to attain the desired ICs, an extra impulsion of value $g_{0l}\Delta t/2$ will occur at the beginning of the signals. Thus, since the ICs are correct, an error is induced at the instant $t = \Delta t$ due to the unwanted residual impulsion mentioned before. If g_{0l} is forced to be 0, an impulsion of the same value will be in default, before $t = 0$, to obtain exactly the desired ICs. An intermediate solution is dividing g_{0l} by 2, doing the prescribed impulsion to the system, but with a time delay of $\Delta t/2$. The effect of any of the three possibilities discussed here is shown in the numerical examples.

Fortunately, it is possible to see that the resulting error decreases with the sampling time Δt, making this error completely controllable.

It is also interesting to note that this error is not present in the methodology proposed for single-dof systems, because in that case it is possible to impose that the previous transient force finishes before $t = 0$. This is not possible for mdof systems, where the previous transient forces are obtained through the product, in the frequency domain, of the inverse FRF matrix of the system by the responses shifted and scaled to give the desired ICs.

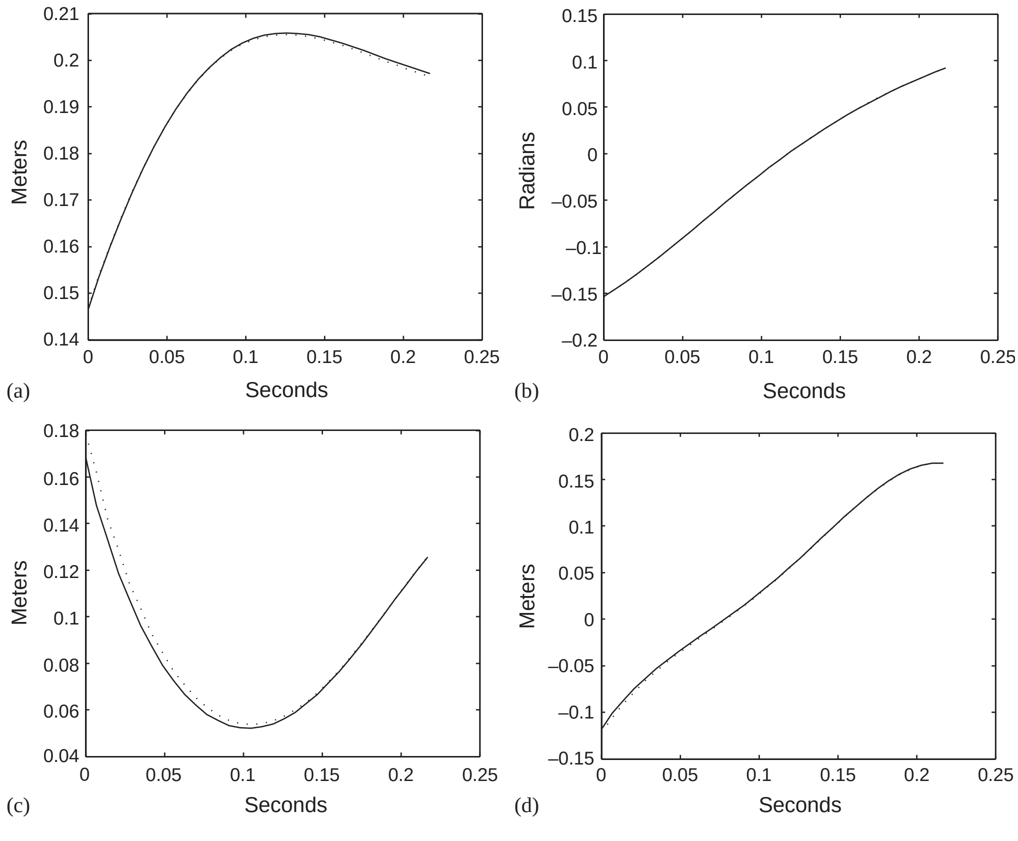

Fig. 7. Detail of responses for example 2: (—) proposed method; (···) Newmark integration. (a) x_1, (b) x_2, (c) x_3, (d) x_4.

4. Numerical examples

To illustrate the proposed methodology, the response of a four dof system subjected to a transient perturbation was calculated using the proposed method, and compared to the numerical step-by-step Newmark integration solution [6].

The matrix representation of this system is

$$[M]\ddot{x} + [C]\dot{x} + [K]x = F, \tag{18}$$

where $[M]$, $[C]$ and $[K]$ are the inertia, damping and stiffness matrices, with the following values:

$$[M] = \begin{bmatrix} 1461.8 & 0 & 0 & 0 \\ 0 & 2176.3 & 0 & 0 \\ 0 & 0 & 10.0 & 0 \\ 0 & 0 & 0 & 10.0 \end{bmatrix}, \tag{19}$$

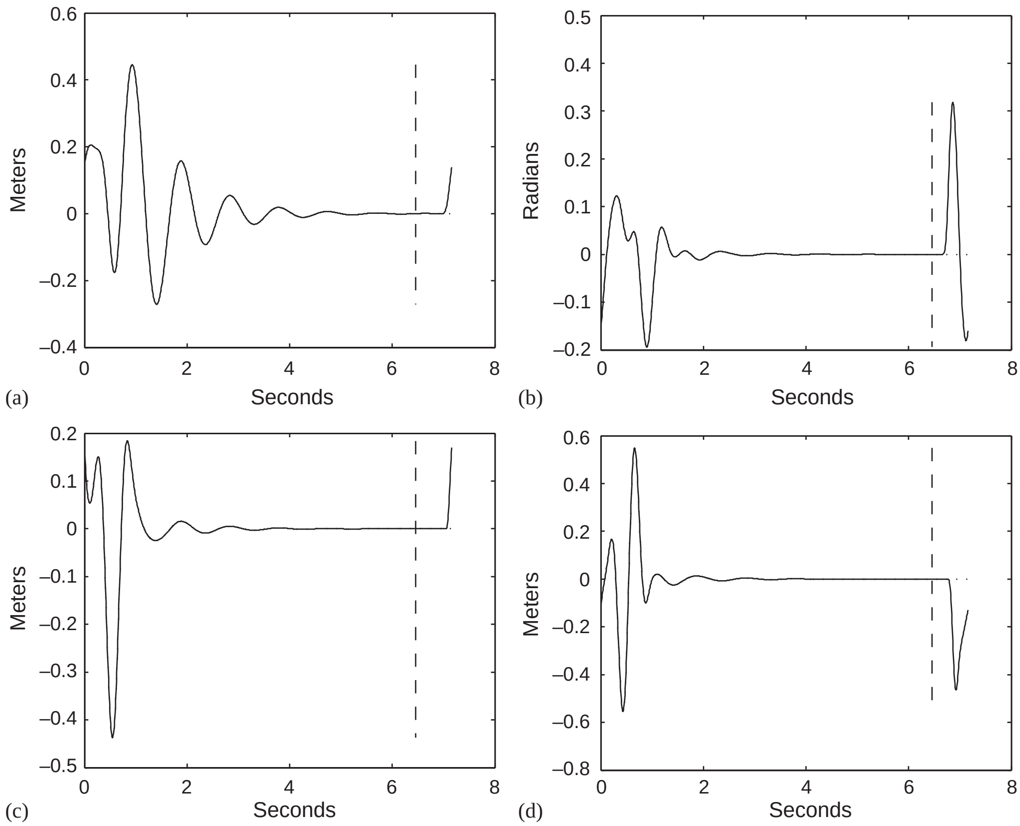

Fig. 8. Responses for example 3: (—) proposed method; (···) Newmark integration. (a) x_1, (b) x_2, (c) x_3, (d) x_4.

$$[K] = 10^5 \times \begin{bmatrix} 0.7295 & 0.1558 & -0.3502 & -0.3793 \\ 0.1558 & 3.4564 & 0.4808 & -0.6365 \\ -0.3502 & 0.4808 & 3.8518 & 0.0000 \\ -0.3793 & -0.6365 & 0.0000 & 4.1728 \end{bmatrix}, \tag{20}$$

$$[C] = 10^4 \times \begin{bmatrix} 0.3648 & 0.0779 & -0.1751 & -0.1897 \\ 0.0779 & 1.7282 & 0.2404 & -0.3183 \\ -0.1751 & 0.2404 & 1.9259 & 0.0000 \\ -0.1897 & -0.3183 & 0.0000 & 2.0864 \end{bmatrix}. \tag{21}$$

These are the matrices of the half-car model shown in Fig. 2, with constant parameters given in Table 1. The damping matrix $[C]$ is proportional to the stiffness matrix $[K]$.

The chosen previous transient (shown in Fig. 1) is the smooth sinusoidal transient described earlier.

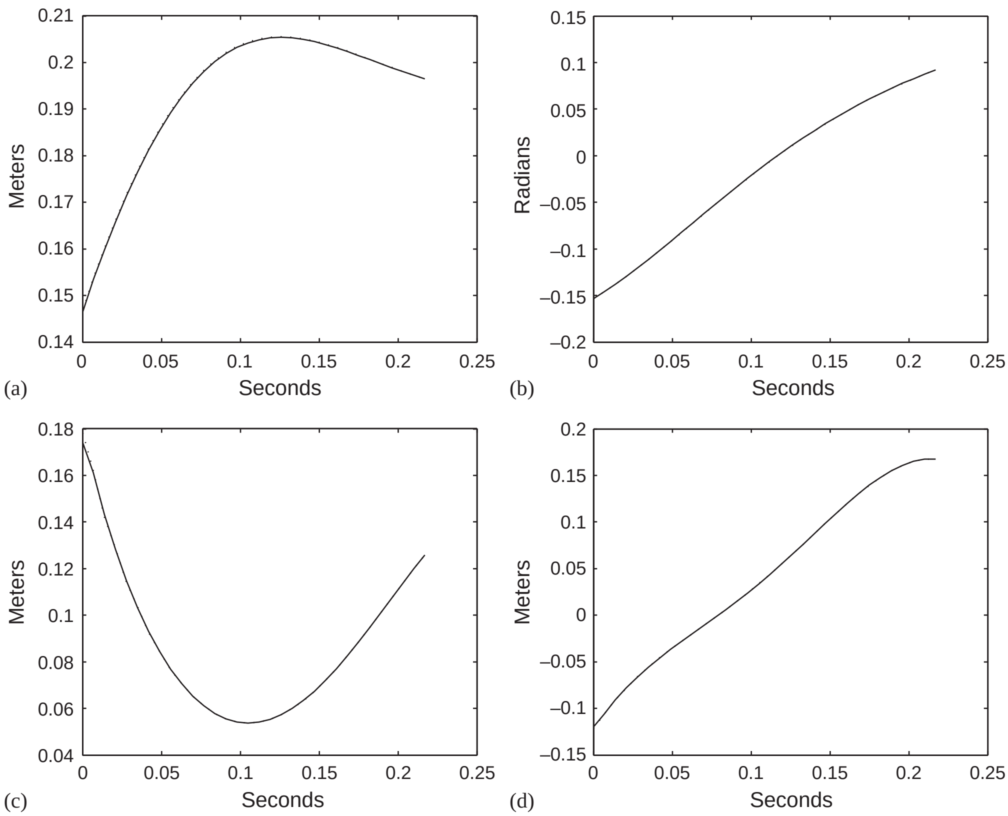

Fig. 9. Detail of responses for example 3: (—) proposed method; (· · ·) Newmark integration. (a) x_1, (b) x_2, (c) x_3, (d) x_4.

The transient excitation forces, in this example, are shown in Fig. 3. They were calculated using the following equation:

$$f_{ij} = \begin{cases} 10^4 i^2 \sin\left(\frac{i\pi j}{n_t}\left(1+\sin\left(\frac{2\pi j}{n_t-\pi/2}\right)\right)\right), & j=0,\ldots,n_t, \\ 0, & j=n_t+1,\ldots,N-1, \end{cases} \tag{22}$$

where $i = 1,\ldots,M$.

The proposed method is suitable for any transient excitation. This choice was arbitrary.

The desired ICs u_0 and $\dot{u}_0$ can be imposed for each dof. A list of discrete relationships $x_0/\dot{x}_0$ is built with the responses obtained by applying the previous transient force (Fig. 1) to all dofs of the system, and the best approximation is automatically chosen. As said in step 4 of the algorithm (single or multiple dof), the relationship between displacement and velocity of the desired response and the response resulting from the previous transient must be the same. However, due to discretization, the instant at which this condition exactly happens is not always accessible. To quantify this error, an index ec_i relating the difference between the coefficients obtained for displacements and velocities, normalized by those obtained for displacements, is built. For each

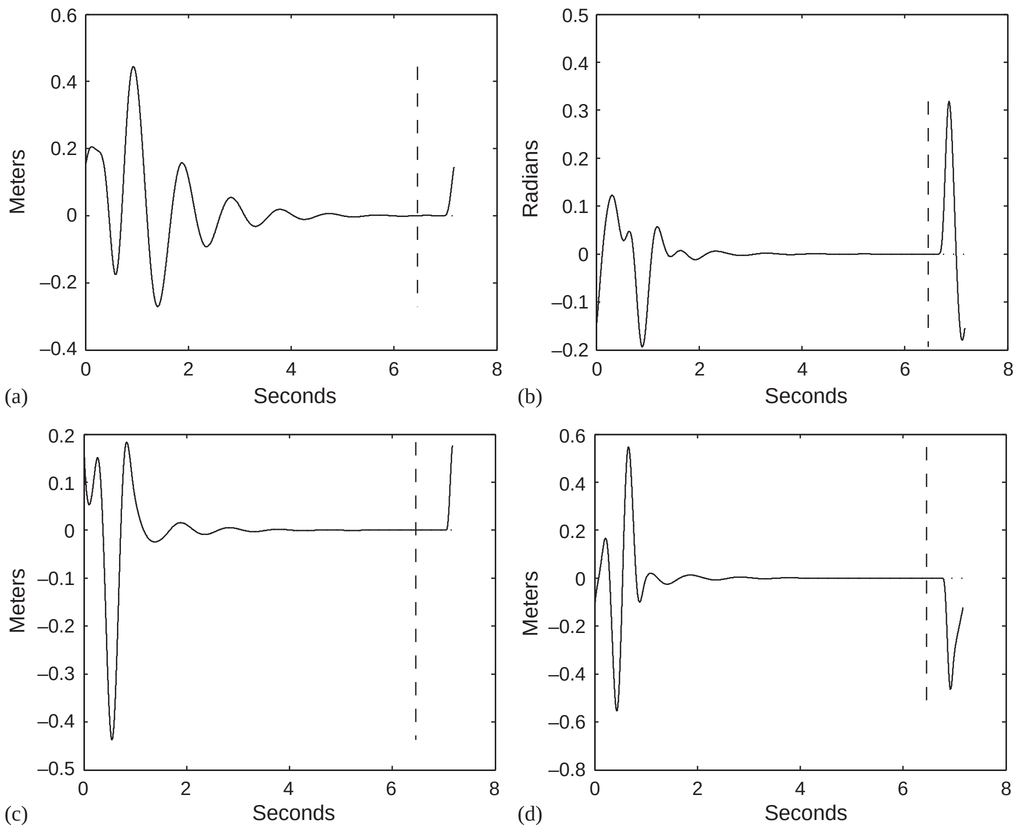

Fig. 10. Responses for example 4: (—) proposed method; (···) Newmark integration. (a) x_1, (b) x_2, (c) x_3, (d) x_4.

dof i, it is given by the equation

$$\mathrm{ec}_i = \left|\frac{\mathrm{Cd}_i - \mathrm{Cv}_i}{\mathrm{Cd}_i}\right| \times 100, \quad i = 1, \ldots, M, \tag{23}$$

where

$$\mathrm{Cd}_i = \frac{u_{i0}}{x_{i0}} \quad \text{and} \quad \mathrm{Cv}_i = \frac{\dot{u}_{i0}}{\dot{x}_{i0}}, \quad i = 1, \ldots, M. \tag{24}$$

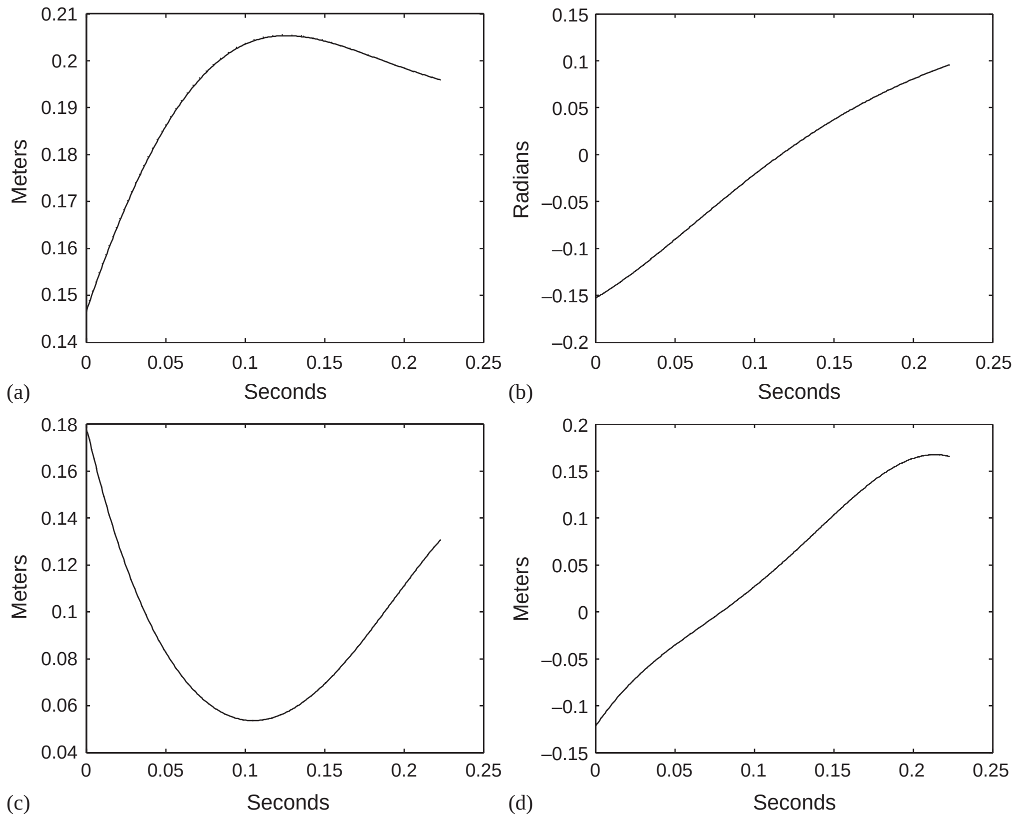

Fig. 11. Detail of responses for example 4: (—) proposed method; (···) Newmark integration. (a) x_1, (b) x_2, (c) x_3, (d) x_4.

Table 2
Errors computed in examples 1–4 (exact IC ratios chosen at step 4 of the algorithm)

	$\overline{\mathrm{ec}}$ (%)	e_d (%)	e_r (%)
Example 1	1.1054e − 12	3.9407e − 1	3.0545e − 3
Example 2	1.1054e − 12	2.2671	3.4396e − 3
Example 3	1.1054e − 12	1.0013	1.8543e − 4
Example 4	1.3954e − 10	1.1015e − 1	9.0571e − 6

For analysis purposes, in the numerical examples the mean value of ec_i for all dofs, called $\bar{\mathrm{ec}}$, is shown:

$$\bar{\mathrm{ec}} = \frac{1}{M}\sum_{i=1}^{M} \mathrm{ec}_i. \tag{25}$$

Tolerance $\mathrm{ec}_{\max}$ is defined as the maximal accepted error $\mathrm{ec}_i \leqslant \mathrm{ec}_{\max}$, $i = 1, \ldots, M$. This index is a good indicator of the compatibility of the signal discretization with the desired accuracy for the ICs, but it is not able, by itself, to clearly show the error in the obtained responses. Therefore, two additional error indexes are used: the mean absolute weighted percentage error in the displacement ICs, calculated by

$$e_d = \frac{1}{M}\sum_{l=1}^{M}\left|\frac{x_{0l} - u_{0l}}{u_{0l}}\right| \times 100 \tag{26}$$

and the weighted mean root mean square (rms) error of the difference between the responses obtained by the proposed method and by the Newmark method,

$$e_r = \frac{1}{M}\sum_{l=1}^{M}\frac{\mathrm{rms}(x_l - X_l)}{\mathrm{rms}(X_l)} \times 100, \tag{27}$$

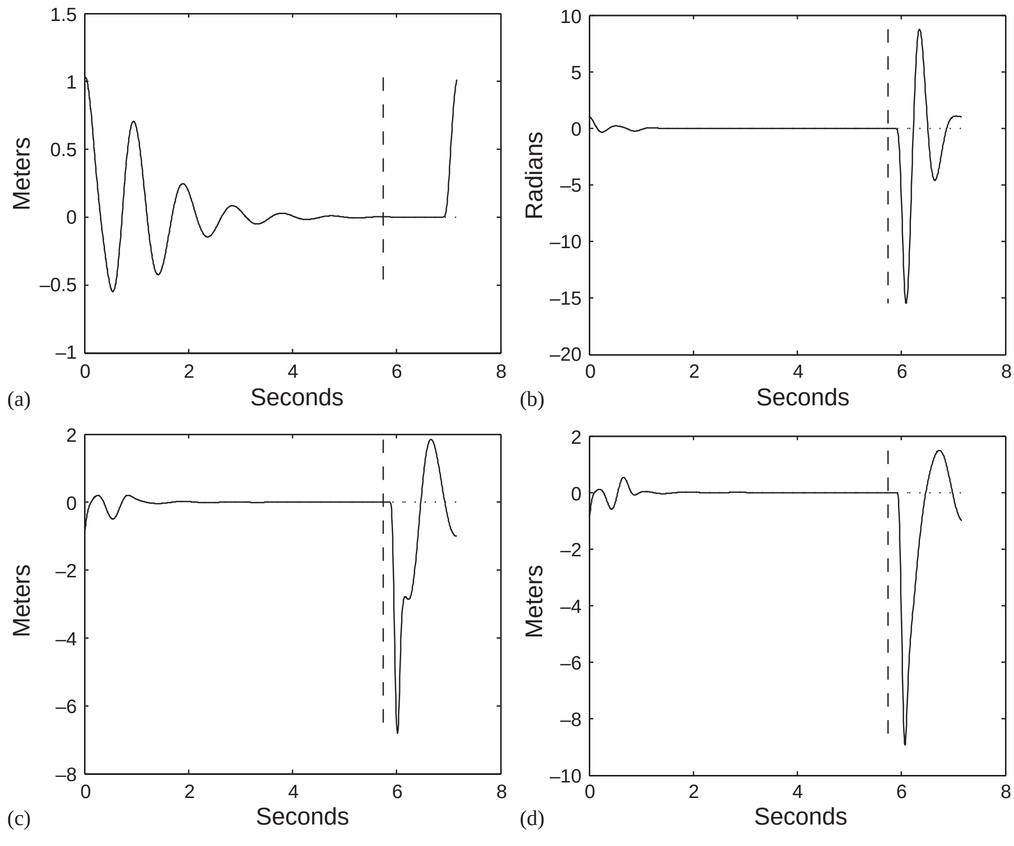

Fig. 12. Responses for example 5: (—) proposed method; (···) Newmark integration. (a) x_1, (b) x_2, (c) x_3, (d) x_4.

where x_l is the response computed by the proposed method and X_l is the response computed by the Newmark method [6].

The procedure is entirely automatic. Given the desired ICs, the system computes, scales and shifts conveniently the previous transient forces, and determines the system responses with ICs as close as possible to the specified ICs within a given tolerance.

Six numerical examples are computed with the four-dof model presented earlier. First, three examples are computed with ICs that exactly match accessible values, thus avoiding problems of accuracy due to time discretization. For these examples, the time sampling was $\Delta t = 7\,\mathrm{ms}$, and $N = 1024$ points. The only difference between these three cases is the choice for the previous transient forces at $t = 0$. Initially, the calculated previous transient force is maintained at $t = 0$. Fig. 4 shows the responses obtained by the proposed method, superposed with those obtained by a Newmark procedure, for the four dofs. The vertical line at the end of these plots limits the region that can be perturbed by the previous transient, in the Fourier solution. The response past this line should be ignored. Fig. 5 presents a zoom in the beginning of the signals, showing the slight difference between solutions obtained by the two procedures.

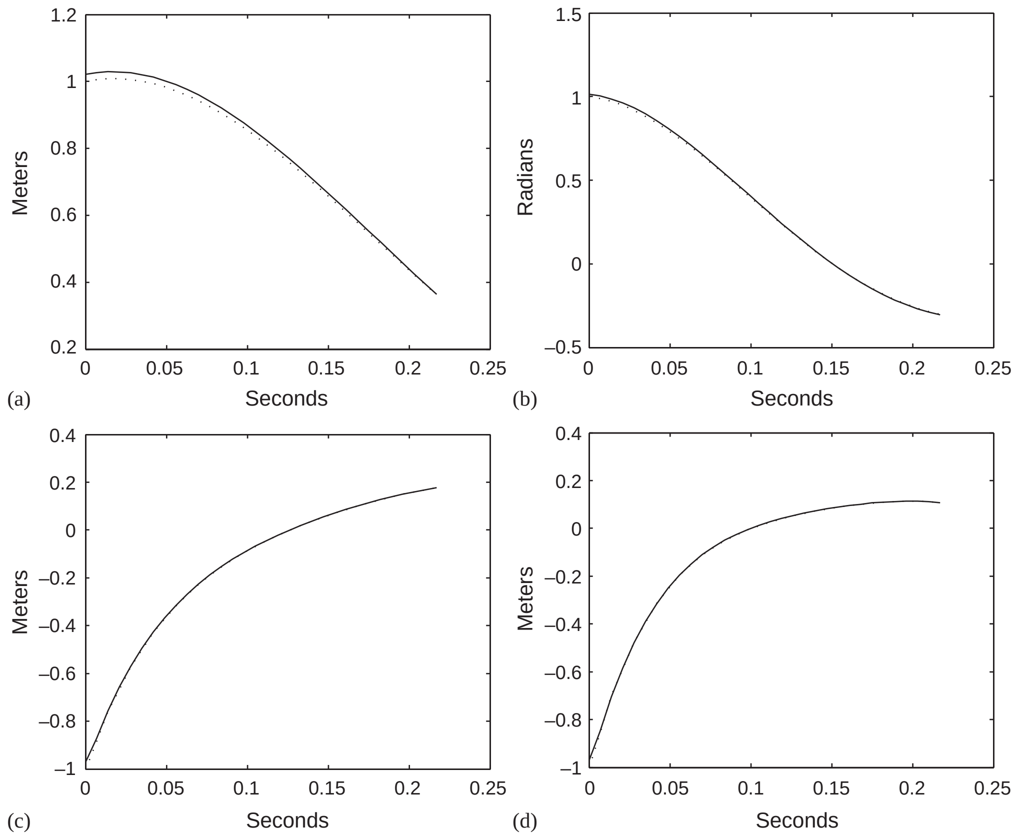

Fig. 13. Detail of responses for example 5: (—) proposed method; (···) Newmark integration. (a) x_1, (b) x_2, (c) x_3, (d) x_4.

In the second case, the previous transient force is set to be 0 at $t = 0$. Figs. 6 and 7 show the superposed results for the full time range and zoomed in, respectively.

The third example is similar to the two formerly presented, except for the value of the previous transient forces at $t = 0$, which is set to half the calculated value. The results are shown in Figs. 8 and 9.

In these examples, the only sources of error are the numerical precision, the leakage of the DFTs, the approximate ODE solution computed by the step-by-step integration method (Newmark), and the residual impulsion $g_{0l}\Delta t/2$ due to discretization, as discussed earlier.

The fourth example is similar to the third, but the time sampling is divided by 8, with $\Delta t = 0.875$ ms. In order to preserve the total time, the number of points is multiplied by 8, i.e., $N = 8192$. Increasing the sampling rate increases the accuracy of the responses, as can be observed in Figs. 10 and 11.

The three error coefficients described earlier are computed for the four cases, and are presented in Table 2. The comparison of these errors for the three first examples show that the third option seems to be the most interesting. The fourth example shows that increasing the sampling frequency increases the accuracy, which means that the error is controllable.

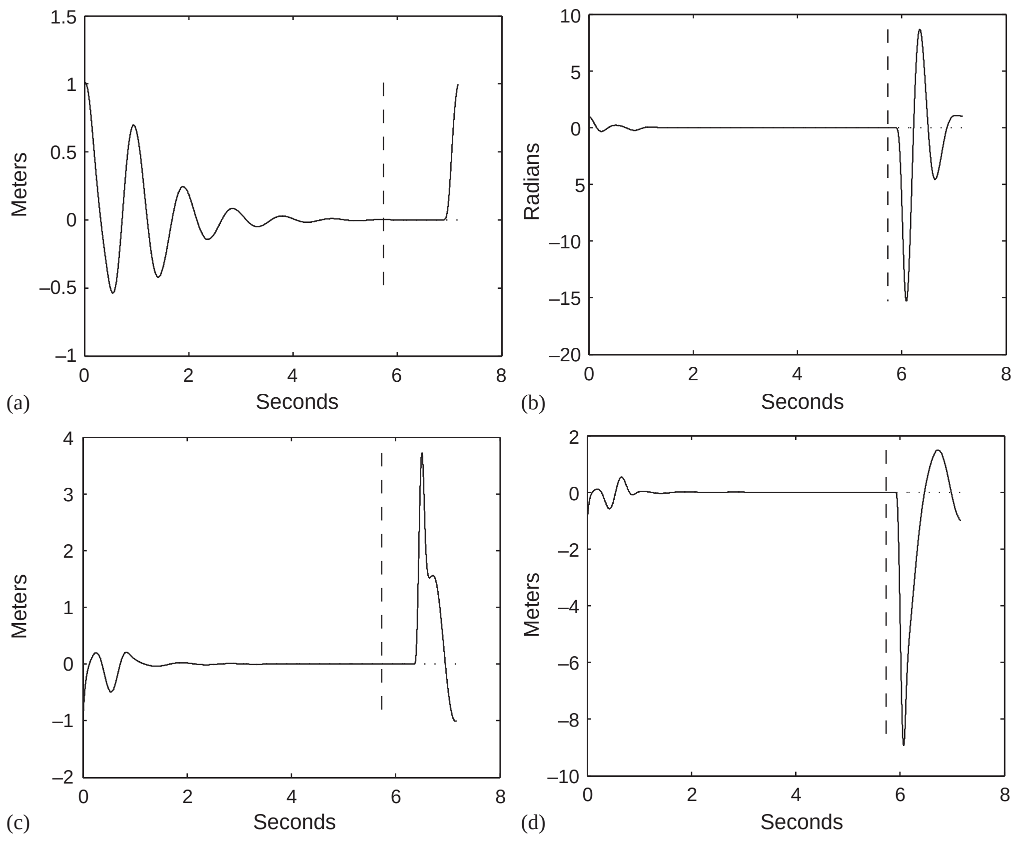

Fig. 14. Responses for example 6: (—) proposed method; (···) Newmark integration. (a) x_1, (b) x_2, (c) x_3, (d) x_4.

A fifth example is computed with similar conditions as the third, but with ICs not choosen to be compatible with the discretization. The desired ICs were $[1, 1, -1, -1]$ for displacements, and $[1, -1, 1, -1]$ for velocities. The results are shown in Figs. 12 and 13.

In order to show the improvement brought by increasing the sampling frequency, the same ICs are searched with the same sampling time of the fourth example. Responses are shown in Figs. 14 and 15.

The error coefficients computed for the two last examples are shown in Table 3.

The FFT allows significant time savings. For the last example presented, the computation time of the response by the proposed method was 2.6 s, and for the Newmark method it was 21.8 s.

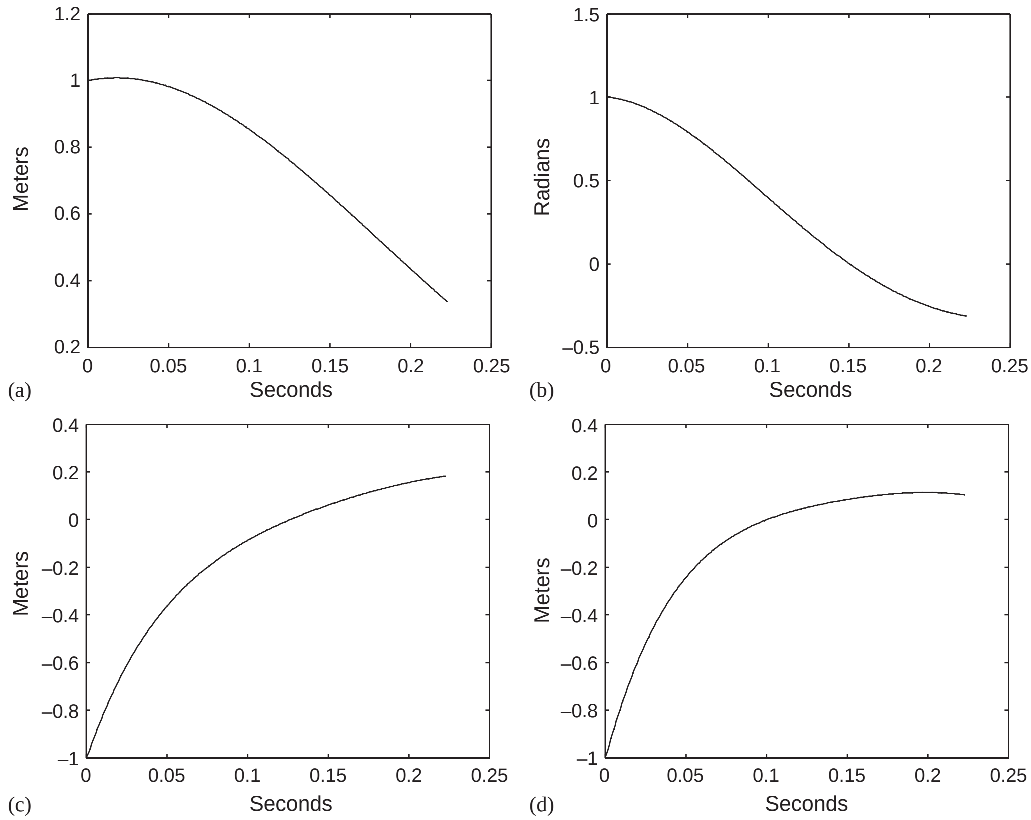

Fig. 15. Detail of responses for example 6: (—) proposed method; (···) Newmark integration. (a) x_1, (b) x_2, (c) x_3, (d) x_4.

Table 3
Errors computed in examples 5 and 6 (approximate IC ratios chosen at step 4 of the algorithm)

	$\bar{ec}$ (%)	e_d (%)	e_r (%)
Example 5	2.5667	2.4643	1.6658
Example 6	2.4878e − 1	1.6870e − 1	1.1734e − 4

Including the time used to determine the previous transient force, needed only once for each new IC set, the time needed for computing the responses using the FFT, as proposed, is approximately three times the time mentioned above, but still significantly smaller than the computing time with the Newmark technique.

5. Conclusions

It was shown that it is possible to calculate the response of damped linear systems to transient excitations with arbitrary initial conditions using the DFT. It is sufficient that the observation window used in the DFT is long enough, so that the transient response nearly vanishes within it. The initial conditions are introduced by a previous artificial transient excitation placed in the tail of the true transient input vector.

The use of FFT algorithms allows a faster computation of the dynamic responses compared with numerical step-by-step integration methods. The method is particularly suited for use with modeling methods where the system equations are in the frequency domain, and it is not possible to obtain a constant-parameter system of ordinary differential equations, as it is the case with spectral methods and boundary element models.

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